Hamiltonian

Let us Consider:

(i) a Conservative system so that potential energy is a function of Co-ordinates only and not that of velocities.

(ii) Constraints do not change with time.

(iii) L Can be written as $L(q_j, \dot{q}_j)$

the total time derivative of $L(q_j, \dot{q}_j)$ will be

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j} \cdot \frac{dq_j}{dt} + \sum_j \frac{\partial L}{\partial \dot{q}_j} \cdot \frac{d\dot{q}_j}{dt}$$

Putting, $\frac{\partial L}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)$

from Lagrange's equation, we get

$$\frac{dL}{dt} = \sum_j \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \dot{q}_j + \frac{\partial L}{\partial \dot{q}_j} \cdot \frac{d\dot{q}_j}{dt}\right]$$

$$= \sum_j \frac{d}{dt}\left(\dot{q}_j \cdot \frac{\partial L}{\partial \dot{q}_j}\right)$$

$$= \sum_j \frac{d}{dt}\left(\dot{q}_j \cdot \frac{\partial T}{\partial \dot{q}_j}\right)$$

because for a conservative system

$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$ as $\frac{\partial V}{\partial \dot{q}_j} = 0$, Also $p_j = \frac{\partial T}{\partial \dot{q}_j}$

Thus,

$$\frac{dL}{dt} = \sum_j \frac{d}{dt}(\dot{q}_j p_j) = 0$$

$$\frac{d}{dt}\left(\sum_j \dot{q}_j p_j - L\right) = 0$$

$$\sum_j \dot{q}_j p_j - L = \text{Constant} = J$$

The constant of motion, J, is one

of the first integrals of equations of motion and is called Jacobi's integral of the system.

The quantity $\left(\sum_j p_j \dot{q}_j - L\right)$ is a constant of motion with the condition that L does not involve time explicity. This constant was designated by a letter H, i.e.

$$H = \sum_j p_j \dot{q}_j - L(q_j, \dot{q}_j)$$

We recognise now H as Hamiltonian and assign to it a basis of (q_j, p_j) set, i.e.

$$H = H(q_j, p_j) = \sum_j p_j \dot{q}_j - L(q_j, \dot{q}_j)$$

If H does not involve time, it is said to be a constant of motion. When H represents the total energy E. It is possible that H may be a constant of motion but not the total energy.

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Hamilton's equations of motion

Hamiltonian is then to be regarded in general, as a function of the position co-ordinates q_j, the momenta p_j, and the time t, i.e.

$$H = H(q_j, p_j, t)$$

Then differential of H gives

$$dH = \sum_j \frac{\partial H}{\partial q_j} dq_j + \sum_j \frac{\partial H}{\partial p_j} \cdot dp_j + \frac{\partial H}{\partial t} \cdot dt \quad (1)$$

Further, H is defined as

$$H = \sum_j p_j \dot{q}_j - L$$

So that $dH = \sum \dot{q}_j \cdot dp_j + \sum p_j \cdot d\dot{q}_j - dL \quad (2)$

But Lagrangian is

$$L = L(q_j, \dot{q}_j, t)$$

So that, $$dL = \sum_j \frac{\partial L}{\partial q_j} \cdot dq_j + \sum_j \frac{\partial L}{\partial \dot{q}_j} \cdot d\dot{q}_j + \frac{\partial L}{\partial t} \cdot dt \quad (3)$$

Putting the value of dL from eq. 3 in eq. 2, we get

$$dH = \sum_j \dot{q}_j\, dp_j + \sum_j p_j\, d\dot{q}_j - \sum_j \frac{\partial L}{\partial q_j}\, dq_j - \sum_j \frac{\partial L}{\partial \dot{q}_j}\, d\dot{q}_j - \frac{\partial L}{\partial t}\, dt \quad \text{— (4)}$$

Recognising

$$\frac{\partial L}{\partial \dot{q}_j} = p_j, \quad \text{and} \quad \frac{\partial L}{\partial q_j} = \dot{p}_j$$

and putting these in eq. 4, we get

$$dH = \sum_j \dot{q}_j\, dp_j + \sum_j p_j\, d\dot{q}_j - \sum_j \dot{p}_j\, dq_j - \sum_j p_j\, d\dot{q}_j - \frac{\partial L}{\partial t}\, dt$$

$$= \sum_j \dot{q}_j\, dp_j - \sum_j \dot{p}_j\, dq_j - \frac{\partial L}{\partial t}\, dt \quad \text{— (5)}$$

Comparing coefficients in eq. 5 and eq. 1, we arrive at

$$\left.\begin{aligned} \dot{q}_j &= \frac{\partial H}{\partial p_j} \\ \dot{p}_j &= -\frac{\partial H}{\partial q_j} \end{aligned}\right\} \quad \text{— (6)}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \quad \text{— (7)}$$

Eq. (6) are known as Hamilton's Canonical equations of motion. They are a set of $2n$ first order equations.

To show that if a given co-ordinate is cyclic in Lagrangian, it will also be cyclic in Hamiltonian:

Suppose q_j does not appear in L then from

$$\dot{p}_j = \frac{\partial L}{\partial q_j} = 0$$

we have p_j = Constant

But if p_j is constant, then from eq. (6), we get that

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} = 0$$

predicting that q_j would also not appear in H. If a co-ordinate is cyclic, it would reduces the number of variables by two in Hamiltonian formulation. Suppose q_n is cyclic then p_n will be some constant, say α, so that $H = H(q_1, \ldots q_{n-1}, p_1, \ldots p_{n-1}, \alpha)$. Thus we shall deal with a problem involving only $(2n-2)$ variables. Since

We can solve the problem ignoring q_n, except for the constant of integration α, the name 'ignorable co-ordinate' is justified. It is possible to find q_n, since

$$\dot{q}_n = \frac{\partial H}{\partial p_n} = \frac{\partial H}{\partial \alpha}$$

will provide it.